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CR-SUBMANIFOLDS OF COMPLEX HYPERBOLIC SPACES SATISFYING A BASIC EQUALITY

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Dedicated to Leopold Verstraelen on his fiftieth birthday

ABSTRACT

The first author introduced a Riemannian invariant denoted by δ and proved in [4] that every n-dimensional submanifold of the complex hyperbolic m-space $\mathbb{C}H^m(4c)$ of constant holomorphic sectional curvature $4c < 0$ satisfies a basic inequality

$$\delta \leq \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)c,$$

where H^2 denotes the squared mean curvature of the submanifold. The main purpose of this paper is to completely classify proper CR-submanifolds of complex hyperbolic spaces which satisfy the equality case of this inequality.

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1. Introduction

For a Riemannian n-manifold M^n denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pM^n$, $p \in M^n$. For an orthonormal basis $e_1, \dots, e_n$ of the tangent space T_pM^n, the scalar curvature τ at p is defined by

$$(1.1) \qquad \tau = \sum_{i<j} K(e_i \wedge e_j).$$

For each point $p \in M^n$, let $(\inf K)(p) = \inf \{ K(\pi)\colon \text{plane sections } \pi \subset T_pM^n \}$. Then $\inf K$ is a well-defined function on M^n. Let δ_M denote the difference between the scalar curvature and $\inf K$, i.e.,

$$(1.2) \qquad \delta_M(p) = \tau(p) - \inf K(p).$$

It is obvious that δ_M is a well-defined Riemannian invariant which is trivial when $n \leq 2$ (cf. [3, 4] for details).

For a submanifold M^n in a real space form $R^m(c)$ of constant sectional curvature c, the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature was first established in [3]:

$$(1.3) \qquad \delta_M \leq \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)c,$$

where H^2 denotes the squared mean curvature.

Let M be a submanifold in a Kaehler m-manifold $\tilde{M}$. A subspace $V \subset T_pM$ is called **totally real** if $JV \subset T_p^{\perp}M$, where T_pM and $T_p^{\perp}M$ denote the tangent space and the normal space of M at p, respectively. The submanifold M is called **totally real** if each tangent space of M is totally real. A totally real submanifold M in $\tilde{M}$ is called **Lagrangian** if $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} \tilde{M}$. A submanifold M of $\tilde{M}$ is called a CR-submanifold if there exists on M a differentiable holomorphic distribution $\mathcal{D}$ such that its orthogonal complement $\mathcal{D}^{\perp} \subset TM$ is a totally real distribution [1]. A CR-submanifold is called **proper** if it is neither totally real (i.e., $\mathcal{D}^{\perp} = TM$) nor holomorphic (i.e., $\mathcal{D} = TM$).

It was remarked in [6] that the exact proof of (1.3) given in [3] yields the same inequality for totally real submanifolds in a complex space form of constant holomorphic sectional curvature $4c$, too. Moreover, it was proved in [4] that inequality (1.3) holds for arbitrary submanifolds of the complex hyperbolic m-space $\mathbb{C}H^m(4c)$ of constant holomorphic sectional curvature $4c < 0$.

For simplicity, an n-dimensional submanifold of $\mathbb{C}H^m(4c)$ is said to **satisfy the basic equality** if it satisfies the equality case of (1.3) identically.

Real hypersurfaces of a Kaehler manifold are proper CR-submanifolds. It was proved in [4] that a real hypersurface of a complex hyperbolic m-space with $m \geq 2$ satisfies the basic equality if and only if the real hypersurface is an open portion of a horosphere in a complex hyperbolic plane.

In this paper we completely classify proper CR-submanifolds of complex hyperbolic spaces which satisfy the basic equality. Moreover, we are able to establish the explicit representation of such submanifolds in an anti-de Sitter space time via Hopf's fibration.

2. Preliminaries

Let $\tilde{M}$ be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric $\tilde{g}$. Denote by $\tilde{\nabla}$ the metric connection of $\tilde{M}$ and by $\langle\, ,\, \rangle$ the inner product induced from the metric $\tilde{g}$. A tangent vector X to $\tilde{M}$ is called space-like (respectively, light-like or time-like) if $\langle X, X\rangle > 0$ or $X = 0$ (respectively, if $\langle X, X\rangle = 0$ and $X \neq 0$ or if $\langle X, X\rangle < 0$).

Let M be a submanifold of $\tilde{M}$. If the metric tensor of $\tilde{M}$ induces a pseudo-Riemannian metric (respectively, Riemannian metric) on M, then M is called a pseudo-Riemannian (respectively, Riemannian) submanifold of $\tilde{M}$. Let ∇ denote the metric connection on M with respect to the induced metric.

For vector fields X, Y tangent to the submanifold, we have the equation of Gauss:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

where h is the second fundamental form of M in $\tilde{M}$. The mean curvature vector $\vec{H}$ of the immersion is given by

$$\vec{H} = \frac{1}{n}\operatorname{trace} h.$$

A submanifold is said to be minimal if its mean curvature vector vanishes identically. Denote by D the linear connection induced on the normal bundle $T^{\perp}M$ of M in $\tilde{M}$. For each vector field ξ normal to M, the Weingarten formula is given by

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

where A is the shape operator. It is well-known that the second fundamental form and the shape operator are related by $\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle$.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M of $\tilde{M}$, respectively, and by R^D the curvature tensor of the normal connection D. Then the equations of Gauss and Ricci are given respectively by

$$R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \tag{2.3}$$

and

$$R^D(X,Y;\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + \langle [A_\xi, A_\eta](X), Y\rangle \tag{2.4}$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M.

For the second fundamental form h, we define the covariant derivative $\bar{\nabla} h$ of h with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.5}$$

The equation of Codazzi is given by

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z). \tag{2.6}$$

The Riemann curvature tensor of a complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ takes the form

(2.7)

$$\tilde{R}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\},$$

where J denotes the almost complex structure of $\tilde{M}(4c)$.

3. Statement of Main Theorem

Consider the complex number $(m+1)$-space $\mathbb{C}_1^{m+1}$ endowed with the pseudo-Euclidean metric g_0 given by (for the details, cf. [7, 9])

$$g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^{m} dz_j d\bar{z}_j, \tag{3.1}$$

where $\bar{z}_k$ denotes the complex conjugate of z_k.

On $\mathbb{C}_1^{m+1}$ we define

$$F(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^{m} z_k \bar{w}_k. \tag{3.2}$$

Put

$$H_1^{2m+1}(-1) = \{z = (z_0, z_1, \ldots, z_m) \in \mathbb{C}_1^{m+1} \colon \langle z, z\rangle = -1\}, \tag{3.3}$$

where $\langle\ ,\ \rangle$ denotes the inner product on $\mathbb{C}_1^{m+1}$ induced from g_0. Then $H_1^{2m+1}(-1)$ is a real hypersurface of $\mathbb{C}^{m+1}$ whose tangent space at $z \in H_1^{2m+1}(-1)$ is given by $T_z H_1^{2m+1}(-1) = \{w \in \mathbb{C}^{m+1}\colon \operatorname{Re} F(z,w) = 0\}$. It is known that $H_1^{2m+1}(-1)$ together with the induced metric g is a pseudo-Riemannian manifold of constant sectional curvature -1, which is known as an anti-de Sitter space time.

We put

$$H_1^1 = \{\lambda \in \mathbb{C}\colon \lambda\bar{\lambda} = 1\}.$$

Then we have an H_1^1-action on $H_1^{2m+1}(-1)$ given by $z \mapsto \lambda z$. At each point z in $H_1^{2m+1}(-1)$, the vector iz is tangent to the flow of the action. Since g_0 is Hermitian, we have $\operatorname{Re} g_0(iz, iz) = -1$. Note that the orbit is given by $x(t) = e^{it}z$ and $dx(t)/dt = ix(t)$. Thus the orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/_\sim$, under the identification induced from the action, is the complex hyperbolic space $\mathbb{C}H^m(-4)$ with constant holomorphic sectional curvature -4. The almost complex structure J on $\mathbb{C}H^m(-4)$ is induced from the canonical almost complex structure J on $\mathbb{C}_1^{m+1}$, the multiplication by i, via the totally geodesic fibration:

$$\pi\colon H_1^{2m+1}(-1) \to \mathbb{C}H^m(-4). \tag{3.4}$$

The main result of this paper is the following.

Main Theorem: *Let U be a domain of $\mathbb{C}$ and $\Psi : U \to \mathbb{C}^{m-1}$ be a nonconstant holomorphic curve in $\mathbb{C}^{m-1}$. Define $z\colon \mathbb{R}^2 \times U \to \mathbb{C}_1^{m+1}$ by*

$$z(u,t,w) = \left(-1 - \frac{1}{2}\Psi(w)\bar{\Psi}(w) + iu, -\frac{1}{2}\Psi(w)\bar{\Psi}(w) + iu, \Psi(w)\right) e^{it}. \tag{3.5}$$

Then $\langle z, z\rangle = -1$ and the image $z(\mathbb{R}^2 \times U)$ in H_1^{2m+1} is invariant under the group action of H_1^1. Moreover, away from points where $\Psi'(w) = 0$, the quotient space $z(\mathbb{R}^2 \times U)/_\sim$ is a proper CR-submanifold of $\mathbb{C}H^m(-4)$ which satisfies the basic equality.

Conversely, up to rigid motions of $\mathbb{C}H^m(-4)$, every proper CR-submanifold of $\mathbb{C}H^m(-4)$ satisfying the basic equality is obtained in such way.

Since, up to rigid motions of $\mathbb{C}H^m(-4)$, a horosphere in $\mathbb{C}H^m(-4)$ is a real hypersurface defined by the equation $|z_1 - z_0| = 1$, Theorem 1 can be regarded as a natural extension of a result of [4] which states that a real hypersurface of $\mathbb{C}H^m(-4)$ with $m \geq 2$ satisfies the basic inequality if and only if it is an open part of a horosphere in $\mathbb{C}H^2(-4)$.

4. Some lemmas

First we recall the following result from [2].

LEMMA 1: *Let M be a CR-submanifold of a Kaehler manifold $\tilde{M}$. Denote by $T^\perp M = J\mathcal{D}^\perp \oplus \nu$ the orthogonal decomposition of the normal bundle, where $\mathcal{D}^\perp$ is the totally real distribution and ν a complex subbundle of $T^\perp M$. We have*

$$\langle \nabla_U Z, X\rangle = \langle J(A_{JZ}U), X\rangle, \tag{4.1}$$

$$A_{JZ}W = A_{JW}Z, \tag{4.2}$$

$$A_{J\xi}X = -A_\xi JX, \tag{4.3}$$

for vector fields Z, W in $\mathcal{D}^\perp$, ξ in ν, U in TM and vector field X in the holomorphic distribution $\mathcal{D}$.

We also need the following two results from [4].

LEMMA 2: *Let $x\colon M \to \mathbb{C}H^m(-4)$ be an isometric immersion of a Riemannian n-manifold M ($n \geq 3$) into the complex hyperbolic m-space $\mathbb{C}H^m(-4)$. Then*

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}H^2 - \frac{1}{2}(n+1)(n-2). \tag{4.4}$$

Equality in (4.4) holds at a point $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^\perp M$ such that (a) *the subspace spanned by $e_3, \ldots, e_n$ is totally real,* (b) *$K(e_1 \wedge e_2) = \inf K$ at p, and* (c) *the shape operators $A_r = A_{e_r}$, $r = n+1, \ldots, 2m$ take the following forms:*

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \ldots & 0 \\ h_{12}^r & h_{22}^r & 0 & \ldots & 0 \\ 0 & 0 & \mu_r & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu_r \end{pmatrix}, \quad r = n+1, \ldots, 2m, \tag{4.5}$$

where $\mu_r = h_{11}^r + h_{22}^r$.

LEMMA 3: *Let $x\colon M \to \mathbb{C}H^m(-4)$ be an isometric immersion of a Riemannian n-manifold M ($n \geq 3$) into the complex hyperbolic m-space $\mathbb{C}H^m(-4)$. If there exists a point $p \in M$ such that*

$$\delta_M = \frac{n^2(n-2)}{2(n-1)}H^2 - \frac{1}{2}(n+1)(n-2), \tag{4.6}$$

at p, then $m \geq n-1$. Furthermore, if $m = n-1$ and (4.6) holds identically, then M is a CR-submanifold.

We also need the following lemmas.

LEMMA 4: *Let $x\colon M \to \mathbb{C}H^m(-4)$ be a CR-submanifold with dimension $n \geq 3$. If M satisfies the basic equality, then one of the following three cases must occurs:*

(1) $n = 3$,

(2) *M is a minimal proper CR-submanifold, or*

(3) *M is a totally real submanifold.*

Proof: Assume that M is a CR-submanifold of $\mathbb{C}H^m(-4)$. Then, according to Lemma 2, either M is totally real or $\mathcal{D} =$: Span $\{e_1, e_2\}$ defines the holomorphic distribution, where $\{e_1, \ldots, e_n\}$ is an orthonormal frame field on M mentioned in Lemma 2. We denote by $\mathcal{D}^\perp$ the totally real distribution spanned by $e_3, \ldots, e_n$.

Suppose that $n > 3$ and M is not totally real. Then without loss of generality we may assume $e_2 = Je_1$.

We divide the proof into two cases.

CASE 1: $m = n-1$. In this case, we have $T^\perp M = J\mathcal{D}^\perp$. If we choose e_3 in such way that $Je_3 = e_{n+1}$ which is parallel to the mean curvature vector, then $\mu_r = 0$ for $r = n+2, \ldots, 2m$. Therefore, from (4.2) and (4.5), we obtain $\mu_{n+1} e_4 = A_{Je_3} e_4 = A_{Je_4} e_3 = 0$ which implies that M is minimal.

CASE 2: $m \geq n$. In this case, there is a complex subbundle ν of the normal bundle perpendicular to $J\mathcal{D}^\perp$ such that $T^\perp M = J\mathcal{D}^\perp \oplus \nu$. From (4.3) we know that, for each vector $\xi \in \nu$, we have

$$A_{J\xi} e_1 = -A_\xi e_2, \quad A_{J\xi} e_2 = A_\xi e_1. \tag{4.7}$$

Therefore

$$\langle A_{J\xi} e_1, e_1 \rangle + \langle A_{J\xi} e_2, e_2 \rangle = -\langle A_\xi e_2, e_1 \rangle + \langle A_\xi e_1, e_2 \rangle = 0, \tag{4.8}$$

which implies that the mean curvature vector $\vec{H}$ lies in $J\mathcal{D}^\perp$. Hence, we may choose e_3 such that Je_3 is parallel to $\vec{H}$. Thus, by applying (4.2) again, we also have $\mu_{n+1} e_4 = A_{Je_3} e_4 = A_{Je_4} e_3 = 0$. Therefore, in this case M is also a minimal proper CR-submanifold of $\mathbb{C}H^m(-4)$. ∎

LEMMA 5: *Let $x: M \to \mathbb{C}H^m(-4)$ be a CR-submanifold. If M satisfies the basic equality and $\dim M > 3$, then M is a totally real submanifold.*

Proof: Assume that M is a proper CR-submanifold of $\mathbb{C}H^m(-4)$ which satisfies the basic equality and $\dim M > 3$. Denote by $\mathcal{D} =$ Span $\{e_1, e_2\}$ the holomorphic distribution. Then, by applying (4.1), Lemma 2 and Lemma 4, we have

$$\begin{aligned}\langle \nabla_{e_2} e_1 - \nabla_{e_1} e_2, Z\rangle &= \langle \nabla_{e_1} Z, e_2\rangle - \langle \nabla_{e_2} Z, e_1\rangle \\ &= \langle J(A_{JZ}e_1), e_2\rangle - \langle J(A_{JZ}e_2), e_1\rangle \\ &= \langle h(e_1,e_1) + h(e_2,e_2), JZ\rangle = 0.\end{aligned}$$

Therefore, $[e_2, e_1] \in \mathcal{D}$ and hence $\mathcal{D}$ is integrable. From Lemma 2, we also have $h(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$, i.e., M is mixed totally geodesic. Hence, M is a mixed foliate CR-submanifold of $\mathbb{C}H^m(-4)$ (cf. [1, 2]). The lemma now follows from a theorem of Chen and Wu [8] which states that every mixed foliate CR-submanifold in a complex hyperbolic space is non-proper. ∎

LEMMA 6: *Let $x: M^3 \to \mathbb{C}H^m(-4)$ be a 3-dimensional proper CR-submanifold of $\mathbb{C}H^m(-4)$. If M satisfies the basic equality, then $\vec{H} \in J\mathcal{D}^\perp$.*

Proof: If $m = 2$, there is nothing to prove. So, we assume $m > 2$. Hence, there is a complex subbundle ν of the normal bundle perpendicular to $J\mathcal{D}^\perp$ such that $T^\perp M = J\mathcal{D}^\perp \oplus \nu$.

If $\vec{H} \notin J\mathcal{D}^\perp$, then there exists a nonzero normal vector field $\xi \in \nu$ such that

$$\vec{H} = \alpha J e_3 + \xi, \tag{4.9}$$

where α is a function and $J\mathcal{D}^\perp =$ Span $\{Je_3\}$. Without loss of generality, we may assume $e_2 = Je_1$. As in Case 2 of Lemma 5, we deduce from (4.3) that

$$\langle A_\xi e_1, e_1\rangle + \langle A_\xi e_2, e_2\rangle = \langle A_{J\xi} e_2, e_1\rangle - \langle A_{J\xi} e_1, e_2\rangle = 0. \tag{4.10}$$

Applying (4.10) and Lemma 2, we obtain trace $A_\xi = 0$, which implies $\langle \vec{H}, \xi\rangle = 0$. This is a contradiction. Therefore, $\vec{H}$ lies in $J\mathcal{D}^\perp$. ∎

LEMMA 7: *Let $x: M \to \mathbb{C}H^m(-4)$ be a CR-submanifold with $n = \dim M \geq 3$. If M satisfies the basic equality, then one of the following two cases must occurs:*

(1) *$n = 3$, M is a proper CR-submanifold with $\vec{H} \in J\mathcal{D}^\perp$ and, moreover, the holomorphic distribution $\mathcal{D}$ is non-integrable on every non-empty open subset of M.*

(2) *$n \geq 3$ and M is a totally real submanifold with $\vec{H} \perp J\mathcal{D}^\perp$.*

Proof: Under the hypothesis, if M is a proper CR-submanifold of $\mathbb{C}H^m(-4)$, then Lemma 5 implies that either $n = 3$ or M is totally real.

If n is 3 and M is not totally real, then Lemma 6 implies that $\vec{H} \in J\mathcal{D}^\perp$. In this case, we deduce from (4.3) that the holomorphic distribution of M is non-integrable on every non-empty open subset of M.

If M is totally real, we have $\vec{H} \perp J\mathcal{D}^\perp$, according to Theorem 1.2 of [5]. ∎

LEMMA 8: *Let $x\colon M^3 \to \mathbb{C}H^m(-4)$ be a 3-dimensional proper CR-submanifold of $\mathbb{C}H^m(-4)$. If M satisfies the basic equality, then M has parallel mean curvature vector, i.e., $D\vec{H} = 0$.*

Proof: Under the hypothesis, we have $\vec{H} \in J\mathcal{D}^\perp$ according to Lemma 6. Thus, Lemma 2 implies $h(X, e_3) \in J\mathcal{D}^\perp$ for any X tangent to M. Hence, using

$$-A_{Je_3}X + D_X(Je_3) = \tilde{\nabla}_X(Je_3) = J(\nabla_X e_3) + h(X, e_3),$$

we obtain $D_X(Je_3) \in J\mathcal{D}^\perp$ for any $X \in TM$. Since $J\mathcal{D}^\perp$ is of rank one and Je_3 is of unit length, this yields $D(Je_3) = 0$. Thus, Je_3 is a parallel normal vector field. Since $\vec{H}$ is parallel to Je_3, it suffices to prove that the mean curvature vector field has constant length.

If $m = 2$, our assumption and Theorem 6 of [4] implies that M is an open part of a horosphere in $\mathbb{C}H^2(-4)$. In this case, by a direct computation we have $\vec{H} = \frac{4}{3}Je_3$. Therefore, M has parallel mean curvature vector.

If $m \geq 3$, then $T^\perp M = J\mathcal{D}^\perp \oplus \nu$ for some complex subbundle ν. Let ξ be a unit vector in ν and X, Y vectors tangent to M. Then we have

$$\tilde{R}(X, Y; Je_3, \xi) = R^D(X, Y; Je_3, \xi) = 0 \tag{4.11}$$

by virtue of (2.7) and $D(Je_3) = 0$. Hence, the equation of Ricci yields $[A_{Je_3}, A_\xi] = 0$ for any $\xi \in \nu$. By Lemma 2, this implies that with respect to a suitable orthonormal frame field with $e_2 = Je_1$ and $e_4 = Je_3$ the shape operators of M in $\mathbb{C}H^m(-4)$ either take the form:

$$A_{Je_3} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2a \end{pmatrix}, \quad A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 \\ h^r_{12} & -h^r_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r = 5, \ldots, 2m, \tag{4.12}$$

or take the form:

$$(4.13)\qquad A_{Je_3} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad A_r = \begin{pmatrix} h^r_{11} & 0 & 0 \\ 0 & -h^r_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r = 5, \ldots, 2m,$$

where $a \neq b$ and $a + b = \mu$.

If the shape operators take the form of (4.13), then, for any $\xi \in \nu$, (4.3) yields $A_{J\xi}e_1 = -A_\xi e_2$. Combining this with (4.13) we obtain $A_5 = \cdots = A_{2m} = 0$. Since Je_3 is a parallel normal vector field, this implies that M is contained in a totally geodesic complex hyperbolic plane $\mathbb{C}H^2(-4)$ of $\mathbb{C}H^m(-4)$. Thus, our assumption implies that M is an open part of a horosphere. Therefore, $\vec{H} = \frac{4}{3}Je_3$ which is a parallel normal vector field.

If the shape operators take the form of (4.12), there exists an orthonormal frame field $\{e_1, e_2, \ldots, e_{2m}\}$ such that the second fundamental form satisfies

$$(4.14)\qquad \begin{aligned} &h(e_1, e_1) = aJe_3 + \phi\xi, \quad h(e_1, e_2) = \phi J\xi, \quad h(e_1, e_3) = 0, \\ &h(e_2, e_2) = aJe_3 - \phi\xi, \quad h(e_2, e_3) = 0, \quad h(e_3, e_3) = 2aJe_3, \end{aligned}$$

where ϕ is a function and ξ is in ν.

Using (4.14) and $D(Je_3) = 0$, we get

$$(4.15)\qquad \begin{aligned} &(\bar{\nabla}_{e_1} h)(e_3, e_3) = 2(e_1 a)Je_3, \\ &(\bar{\nabla}_{e_3} h)(e_1, e_3) = -2a\langle \nabla_{e_3} e_1, e_3\rangle Je_3 - \langle \nabla_{e_3} e_3, e_1\rangle (aJe_3 + \phi\xi) - \phi\langle \nabla_{e_3} e_3, e_2\rangle J\xi. \end{aligned}$$

The equation of Codazzi, (2.7) and (4.15) yield

$$(4.16)\qquad \phi\langle \nabla_{e_3} e_3, e_1\rangle = \phi\langle \nabla_{e_3} e_3, e_2\rangle = 0.$$

If $\phi \equiv 0$ on M, $D(Je_3) = 0$ and (4.14) imply that M is contained in a totally geodesic complex hyperbolic plane $\mathbb{C}H^2(-4)$. Thus, M is an open part of a horosphere. Hence, we know that M has parallel mean curvature vector as before.

If $\phi \not\equiv 0$, we put $V = \{p \in M\colon \phi(p) \neq 0\}$. Then (4.16) implies $\nabla_{e_3} e_3 = 0$ on V. Hence, by using $D(Je_3) = 0$, (4.16) and the equation of Codazzi, we obtain $e_1 a = 0$. Similarly, we have $e_2 a = 0$. Therefore, $[e_1, e_2]\, a = 0$.

On the other hand, since $\mathcal{D} =$ Span $\{e_1, e_2\}$ is non-integrable on every non-empty open subset of M according to Lemma 7, the Lie bracket $[e_1, e_2]$ has a non-trivial component in the direction of e_3 at every point in a dense open subset of M. Therefore, the condition $[e_1, e_2]a = 0$ implies that $e_3 a = 0$ by continuity. Consequently, the function a must be a constant. Combining this we obtain

$D(Je_3) = 0$, thus we conclude that the mean curvature vector $\vec{H} = \frac{4}{3}aJe_3$ is parallel in the normal bundle. ∎

A submanifold is said to be **linearly full in** $\mathbb{C}H^m(-4)$ if it does not lie in any totally geodesic complex hypersurface of $\mathbb{C}H^m(-4)$.

LEMMA 9: *Let x: $M^3 \to \mathbb{C}H^m(-4)$ be a linearly full 3-dimensional proper CR-submanifold of $\mathbb{C}H^m(-4)$. If $m \geq 3$ and M satisfies the basic equality, then, with respect to some suitable orthonormal frame field $\{e_1, e_2, \dots, e_{2m}\}$, we have*

$$\begin{aligned} h(e_1,e_1) &= Je_3 + \phi\xi, \quad h(e_1,e_2) = \phi J\xi, \quad h(e_1,e_3) = 0, \\ h(e_2,e_2) &= Je_3 - \phi\xi, \quad h(e_2,e_3) = 0, \quad h(e_3,e_3) = 2Je_3, \end{aligned} \tag{4.17}$$

$$\begin{gathered} \nabla_{e_1}e_1 = \alpha e_2, \quad \nabla_{e_1}e_2 = -\alpha e_1 - e_3, \quad \nabla_{e_1}e_3 = e_2, \\ \nabla_{e_2}e_1 = -\beta e_2 + e_3, \quad \nabla_{e_2}e_2 = -\beta e_1, \quad \nabla_{e_2}e_3 = -e_1, \\ \nabla_{e_3}e_1 = \gamma e_2, \quad \nabla_{e_3}e_2 = -\gamma e_1, \quad \nabla_{e_3}e_3 = 0, \end{gathered} \tag{4.18}$$

where $\alpha, \beta, \gamma, \phi$ are functions such that $\phi \neq 0$ and ξ is in ν. In particular, M has constant squared mean curvature $H^2 = 16/9$.

Proof: Let x: $M^3 \to \mathbb{C}H^m(-4)$ be a linearly full 3-dimensional proper CR-submanifold of $\mathbb{C}H^m(-4)$ satisfying the basic equality. If $m \geq 3$, then from the proof of Lemma 8 we know that, with respect a suitable orthonormal frame field with $e_2 = Je_1$ and $e_4 = Je_3$, the second fundamental form satisfies

$$\begin{aligned} h(e_1,e_1) &= aJe_3 + \phi\xi, \quad h(e_1,e_2) = \phi J\xi, \quad h(e_1,e_3) = 0, \\ h(e_2,e_2) &= aJe_3 - \phi\xi, \quad h(e_2,e_3) = 0, \quad h(e_3,e_3) = 2aJe_3, \end{aligned} \tag{4.19}$$

where ϕ is a nonzero function, a is a constant, and ξ is in ν.

From (4.1) of Lemma 1 and (4.19) we have

$$\begin{aligned} \langle \nabla_{e_1}e_1, e_3\rangle &= -\langle \nabla_{e_1}e_3, e_1\rangle = \langle A_{Je_3}e_1, e_2\rangle = 0, \\ \langle \nabla_{e_1}e_2, e_3\rangle &= -\langle \nabla_{e_1}e_3, e_2\rangle = -\langle A_{Je_3}e_1, e_1\rangle = -a, \end{aligned}$$

which implies

$$\nabla_{e_1}e_1 = \alpha e_2, \quad \nabla_{e_1}e_2 = -\alpha e_1 - ae_3, \quad \nabla_{e_1}e_3 = ae_2, \tag{4.20}$$

for some function α. Similarly, we also have

$$\nabla_{e_2}e_1 = -\beta e_2 + ae_3, \quad \nabla_{e_2}e_2 = -\beta e_1, \quad \nabla_{e_2}e_3 = -ae_1, \tag{4.21}$$

for some function β. Using (4.19), (4.20) and (4.21), we find

$$(\bar{\nabla}_{e_1}h)(e_2,e_3) = a^2Je_3 + a\phi\xi,$$
$$(\bar{\nabla}_{e_2}h)(e_1,e_3) = -a^2Je_3 + a\phi\xi. \tag{4.22}$$

On the other hand, (2.7) yields $\tilde{R}(e_1,e_2,e_3,Je_3) = 2$. Thus, by (4.22) and the equation of Codazzi, we obtain $a^2 = 1$. Replace e_3 by $-e_3$ if necessary; we have $a = 1$. Thus, (4.19) yields (4.17). From (4.20), (4.21) and $a = 1$, we obtain the first six equations in (4.18).

Using (4.1) and (4.19) we get $\langle \nabla_{e_3}e_3, e_i\rangle = \langle J(A_{Je_3}e_3), e_i\rangle = 0$ for $i = 1,2$. Hence, $\nabla_{e_3}e_3 = 0$ which yields the last three equations of (4.18). ∎

5. Proof of the Main Theorem

Let U be a domain of $\mathbb{C}$ and $\Psi\colon U \to \mathbb{C}^{m-1}$ be a nonconstant holomorphic curve in $\mathbb{C}^{m-1}$ such that $\Psi'(w)$ is nowhere zero. Define $z\colon \mathbb{R}^2 \times U \to \mathbb{C}_1^{m+1}$ by

$$z(u,t,w) = \left(-1 - \frac{1}{2}\Psi(w)\bar{\Psi}(w) + iu, -\frac{1}{2}\Psi(w)\bar{\Psi}(w) + iu, \Psi(w)\right)e^{it}. \tag{5.1}$$

Then $\langle z,z\rangle = -1$. Thus, the image $z(\mathbb{R}^2 \times U)$ of $\mathbb{R}^2 \times U$ under z is contained in the anti-de Sitter space time $H_1^{2m+1}(-1)$.

Let $w = x + iy$ denote the standard coordinate of $U \subset \mathbb{C}$. Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}, \quad \frac{\partial}{\partial y} = i\frac{\partial}{\partial w} - i\frac{\partial}{\partial \bar{w}}. \tag{5.2}$$

We obtain from (5.1) and (5.2) that

$$\begin{aligned} z_u &= (i,i,0)e^{it}, \quad z_t = iz, \\ z_x &= (-\frac{1}{2}(\Psi'\bar{\Psi} + \Psi\bar{\Psi}'), -\frac{1}{2}(\Psi'\bar{\Psi} + \Psi\bar{\Psi}'), \Psi')e^{it}, \\ z_y &= i(-\frac{1}{2}(\Psi'\bar{\Psi} - \Psi\bar{\Psi}'), -\frac{1}{2}(\Psi'\bar{\Psi} - \Psi\bar{\Psi}'), \Psi')e^{it}. \end{aligned} \tag{5.3}$$

$$\begin{aligned} &z_{uu} = 0, \quad z_{ut} = iz_u, \quad z_{ux} = z_{uy} = 0, \\ &z_{tt} = -z, \quad z_{tx} = iz_x, \quad z_{ty} = iz_y \\ z_{xx} &= (-\frac{1}{2}(\Psi''\bar{\Psi} + 2\Psi'\bar{\Psi}' + \Psi\bar{\Psi}''), -\frac{1}{2}(\Psi''\bar{\Psi} + 2\Psi'\bar{\Psi}' + \Psi\bar{\Psi}''), \Psi'')e^{it}, \\ z_{xy} &= i(\frac{1}{2}(\Psi\bar{\Psi}'' - \Psi''\bar{\Psi}), \frac{1}{2}(\Psi\bar{\Psi}'' - \Psi''\bar{\Psi}), \Psi'')e^{it}, \\ z_{yy} &= (\frac{1}{2}(\Psi''\bar{\Psi} - 2\Psi'\bar{\Psi}' + \Psi\bar{\Psi}''), \frac{1}{2}(\Psi''\bar{\Psi} - 2\Psi'\bar{\Psi}' + \Psi\bar{\Psi}''), -\Psi'')e^{it}. \end{aligned} \tag{5.4}$$

Let $\lambda = |\Psi'|$ and

$$(5.5)\; E_1 = \frac{1}{\lambda}\left(z_x - \frac{1}{2}i(\Psi\bar{\Psi}' - \Psi'\bar{\Psi})z_u\right), \quad E_2 = \frac{1}{\lambda}\left(z_y - \frac{1}{2}(\Psi\bar{\Psi}' + \Psi'\bar{\Psi})z_u\right),$$
$$E_3 = iz + z_u = z_t + z_u, \quad E_4 = iz = z_t.$$

Then E_1, E_2, E_3, E_4 are orthonormal tangent vector fields such that $E_2 = iE_1$ and iE_3, iE_4 are normal vector fields.

It follows from (5.3), (5.4), (5.5) and a straightforward computation that the second fundamental form $\tilde{h}$ of $z(\mathbb{R}^2 \times U)$ in $\mathbb{C}_1^{m+1}$ satisfies

$$\begin{aligned}
\tilde{h}(E_1,E_1) &= \frac{1}{2\lambda^2}\{(\Psi''\bar{\Psi} + 2\Psi'\bar{\Psi}' + \Psi\bar{\Psi}'')iz_u + (0,0,2\Psi''e^{it})^{\perp}\},\\
\tilde{h}(E_1,E_2) &= \frac{1}{2\lambda^2}\{i(\Psi''\bar{\Psi} - \Psi\bar{\Psi}'')iz_u + (0,0,2i\Psi''e^{it})^{\perp}\},\\
\tilde{h}(E_2,E_2) &= \frac{-1}{2\lambda^2}\{(\Psi''\bar{\Psi} - 2\Psi'\bar{\Psi}' + \Psi\bar{\Psi}'')iz_u + (0,0,2\Psi''e^{it})^{\perp}\},\\
\tilde{h}(E_1,E_3) &= \tilde{h}(E_2,E_3) = \tilde{h}(E_1,E_4) = \tilde{h}(E_2,E_4) = 0,\\
\tilde{h}(E_3,E_3) &= 2iE_3 - iE_4, \quad \tilde{h}(E_3,E_4) = iE_3, \quad \tilde{h}(E_4,E_4) = iE_4,
\end{aligned} \tag{5.6}$$

where $\{\cdots\}^{\perp}$ denotes the normal component of $\{\cdots\}$.

On the other hand, from (5.3), (5.4), (5.5) and (5.6), we have

$$\langle\tilde{h}(E_1,E_1), iE_3\rangle = \langle\tilde{h}(E_1,E_1), iE_4\rangle = 1.$$

Therefore, there exist a function ϕ and a unit normal vector field $\tilde{\xi}$ perpendicular to iE_3 and iE_4 such that

$$\tilde{h}(E_1,E_1) = iE_3 - iE_4 + \phi\tilde{\xi}. \tag{5.7}$$

From (5.6) and (5.7) we obtain

$$(0,0,\Psi''e^{it})^{\perp} = -\frac{1}{2}(\Psi\bar{\Psi}'' + \Psi''\bar{\Psi})iz_u + \lambda^2\phi\tilde{\xi}. \tag{5.8}$$

Similarly, we have

$$\tilde{h}(E_2,E_2) = iE_3 - iE_4 - \phi\tilde{\xi}. \tag{5.9}$$

$$(0,0,i\Psi''e^{it})^{\perp} = \frac{1}{2}i(\Psi\bar{\Psi}'' - \Psi''\bar{\Psi})iz_u + i\lambda^2\phi\tilde{\xi}. \tag{5.10}$$

Since iz is always tangent to $z(\mathbb{R}^2 \times U)$, the image $z(\mathbb{R}^2 \times U)$ in $H_1^{2m+1}(-1)$ is invariant under the group action of H_1^1. Hence, $z(\mathbb{R}^2 \times U)$ is projectable via the Hopf's fibration π: $H_1^{2m+1}(-1) \to \mathbb{C}H^m(-4)$. It is known that the Hopf fibration

π is a Riemannian submersion. The image $\pi(z(\mathbb{R}^2 \times U))$ is a 3-dimensional proper CR-submanifold of $\mathbb{C}H^m(-4)$ whose holomorphic distribution $\mathcal{D}$ is spanned by $\pi_*(E_1), \pi_*(E_2)$ and whose totally real distribution $\mathcal{D}^\perp$ is spanned by $\pi_*(E_3)$.

It follows from (5.5)–(5.10) that the second fundamental form h of $\pi(z(\mathbb{R}^2 \times U))$ in $\mathbb{C}H^m(-4)$ satisfies

$$\begin{aligned}(5.11)\quad & h(e_1,e_1) = Je_3 + \phi\xi, \quad h(e_1,e_2) = \phi J\xi, \quad h(e_2,e_2) = Je_3 - \phi\xi, \\ & h(e_1,e_3) = h(e_2,e_3) = 0, \quad h(e_3,e_3) = 2Je_3,\end{aligned}$$

where $\xi = \pi_*(\tilde{\xi})$ is a normal vector field perpendicular to Je_3, $e_1 = \pi_*(E_1)$ and $e_2 = \pi_*(E_2)$. Therefore, by applying Lemma 2, we conclude that the 3-dimensional proper CR-submanifold $\pi(z(\mathbb{R}^2 \times U))$ in $\mathbb{C}H^m(-4)$ satisfies the basic equality.

Conversely, suppose that M is a proper CR-submanifold of $\mathbb{C}H^m(-4)$ with $\dim M \geq 3$ which satisfies the basic equality. Then, according to Lemma 7, the dimension of M is equal to 3. Moreover, with respect to some suitable orthonormal frame field $\{e_1, e_2, \ldots, e_{2m}\}$, the second fundamental form h and the Riemannian connection ∇ of M satisfy (4.17) and (4.18), respectively. Furthermore, by Lemma 8, we have $D(Je_3) = 0$.

Let $\hat{M} = \pi^{-1}(M)$ denote the inverse image of M via the Hopf fibration π: $H_1^{2m+1} \to \mathbb{C}H^m(-4)$. Then $\hat{M}$ is a principal circle bundle over M with time-like totally geodesic fibers. Let z: $\hat{M} \to H_1^{2m+1}(-1) \subset \mathbb{C}_1^{m+1}$ denote the immersion of $\hat{M}$ in $\mathbb{C}_1^{m+1}$.

Let $\tilde{\nabla}$ and $\hat{\nabla}$ denote the metric connections of $\mathbb{C}_1^{m+1}$ and $H_1^{2m+1}(-1)$, respectively. We denote by X^* the horizontal lift of a tangent vector X of $\mathbb{C}H^m(-4)$ with respect to $\hat{\nabla}$. Then we have (cf. [7, 9])

$$(5.12)\qquad \tilde{\nabla}_{X^*}Y^* = (\nabla_X Y)^* + (h(X,Y))^* + \langle JX, Y\rangle V + \langle X, Y\rangle z,$$

$$(5.13)\qquad \tilde{\nabla}_{X^*}V = \tilde{\nabla}_V X^* = (JX)^*,$$

$$(5.14)\qquad \tilde{\nabla}_V V = -z,$$

for vector fields X, Y tangent to M, where z is the position vector of $\hat{M}$ in $\mathbb{C}_1^{2m+1}$ and $V = iz \in T_z H_1^{2m+1}(-1)$.

Let E_1, E_2, E_3 be the horizontal lifts of e_1, e_2, e_3, respectively and let $E_4 = iz$, and hence $z = iE_4$. Then, from Lemma 9, (5.12), (5.13) and (5.14), we obtain

$$(5.15\text{-a})\qquad \tilde{\nabla}_{E_1}E_1 = \alpha E_2 + iE_3 + \phi\xi^* - iE_4,$$

$$(5.15\text{-b})\qquad \tilde{\nabla}_{E_1}E_2 = -\alpha E_1 - E_3 + \phi i\xi^* + E_4,$$

$$(5.15\text{-c})\qquad \tilde{\nabla}_{E_1}E_3 = \tilde{\nabla}_{E_1}E_4 = E_2,$$

$$\tilde{\nabla}_{E_2}E_1 = -\beta E_2 + E_3 + \phi i\xi^* - E_4, \tag{5.15-d}$$

$$\tilde{\nabla}_{E_2}E_2 = \beta E_1 + iE_3 - \phi\xi^* - iE_4, \tag{5.15-e}$$

$$\tilde{\nabla}_{E_2}E_3 = \tilde{\nabla}_{E_2}E_4 = -E_1, \tag{5.15-f}$$

$$\tilde{\nabla}_{E_3}E_1 = \gamma E_2, \tag{5.15-g}$$

$$\tilde{\nabla}_{E_3}E_2 = -\gamma E_1, \tag{5.15-h}$$

$$\tilde{\nabla}_{E_3}E_3 = 2iE_3 - iE_4, \tag{5.15-i}$$

$$\tilde{\nabla}_{E_3}E_4 = iE_3, \tag{5.15-j}$$

$$\tilde{\nabla}_{E_4}E_1 = E_2, \tag{5.15-k}$$

$$\tilde{\nabla}_{E_4}E_2 = -E_1, \tag{5.15-m}$$

$$\tilde{\nabla}_{E_4}E_3 = iE_3, \tag{5.15-n}$$

$$\tilde{\nabla}_{E_4}E_4 = iE_4. \tag{5.15-o}$$

Equations (5.15-b), (5.15-c), (5.15-d) and (5.15-f)–(5.15-o) imply that the distribution $\mathcal{D}_1$ spanned by $E_1, E_2, E_3 - E_4$ is integrable. The distribution $\mathcal{D}_2$ spanned by E_3 is clearly integrable, since it is of rank one. Hence, there exist coordinates $\{s, t, q, v\}$ such that $\partial/\partial s, \partial/\partial q$ and $\partial/\partial v$ are tangent to integral submanifolds of $\mathcal{D}_1$, $\partial/\partial s = E_3 - E_4$ and $\partial/\partial t = E_3$.

Applying (5.15-c), (5.15-f), (5.15-i), (5.15-j), (5.15-n) and (5.15-o), we get

$$\tilde{\nabla}_{E_1}(E_3 - E_4) = \tilde{\nabla}_{E_2}(E_3 - E_4) = \tilde{\nabla}_{E_3 - E_4}(E_3 - E_4) = 0.$$

Hence, along each integral submanifold of $\mathcal{D}_1$, $Z =: E_3 - E_4$ is a constant light-like vector in $\mathbb{C}_1^{m+1}$. Moreover, from (5.15-i) and (5.15-j), we have $\tilde{\nabla}_{E_3}Z = iZ$. Since $E_3 = \partial/\partial t$, we get $\partial Z/\partial t = iZ$. Solving this differential equation yields

$$Z = e^{it}Z_0 \quad \text{on } \hat{M}, \tag{5.16}$$

where Z_0 is a light-like constant vector. Without loss of generality, we may assume $Z_0 = (i, i, 0, \ldots, 0) \in \mathbb{C}_1^{m+1}$.

Let M_1 be an integral submanifold of $\mathcal{D}_1$. Without loss of generality, we may assume that M_1 is defined by $t = 0$. From (5.15-a)–(5.15-f), we obtain

$$\tilde{\nabla}_{E_1}E_1 = \alpha E_2 + \phi\xi^* + iZ, \tag{5.17-a}$$

$$\tilde{\nabla}_{E_1}E_2 = -\alpha E_1 + i\phi\xi^* - Z, \tag{5.17-b}$$

$$\tilde{\nabla}_{E_1}Z = \tilde{\nabla}_{E_2}Z = 0, \tag{5.17-c}$$

$$\tilde{\nabla}_{E_2}E_1 = -\beta E_2 + i\phi\xi^* + Z, \tag{5.17-d}$$

$$\tilde{\nabla}_{E_2}E_2 = \beta E_1 - \phi\xi^* + iZ, \tag{5.17-e}$$

$$\tilde{\nabla}_{E_1}\xi^* = -\phi E_1 + \nabla^{\perp}_{E_1}\xi^*, \tag{5.17-f}$$

$$\tilde{\nabla}_Z E_1 = (\gamma - 1)E_2, \tag{5.17-g}$$

$$\tilde{\nabla}_Z E_2 = (1 - \gamma)E_1, \tag{5.17-h}$$

$$\tilde{\nabla}_Z Z = 0, \tag{5.17-i}$$

$$\tilde{\nabla}_{E_2}\xi^* = \phi E_2 + \nabla^{\perp}_{E_2}\xi^*, \tag{5.17-j}$$

$$\tilde{\nabla}_Z\xi^* = \nabla^{\perp}_Z\xi^*, \tag{5.17-k}$$

where $\nabla^{\perp}$ denotes the normal connection of M_1 in $\mathbb{C}_1^{2m+1}$.

Along M_1 we have that

$$\begin{aligned}\langle Z_0, z\rangle &= \langle E_3 - E_4, z\rangle = \langle E_3 - E_4, -iE_4\rangle = 0,\\ \langle Z_0, iz\rangle &= \langle E_3 - E_4, iz\rangle = \langle E_3 - E_4, E_4\rangle = -1,\\ \langle Z_0, Z_0\rangle &= \langle Z_0, E_1\rangle = \langle Z_0, E_2\rangle = \langle Z_0, \tilde{\nabla}_X Y\rangle = 0,\end{aligned} \tag{5.18}$$

where $X, Y \in$ Span $\{Z_0, E_1, E_2\}$. Since Z_0 is a constant vector along M_1, the above equations imply that M_1 lies in a complex hyperplane which is parallel to $\{Z_0\}^{\perp}$. Since $\{Z_0\}^{\perp}$ is spanned by

$$(i, i, 0, \dots, 0),\ (0, 0, 1, 0. \dots, 0), \dots, (0, 0, 0, \dots, 0, 1),$$

it follows that we can write

$$\begin{aligned} z(s, 0, w) =& f(s, w)(i, i, 0, \dots, 0) + c(1, -1, 0, \dots, 0)\\ &+ (0, 0, \Psi_1(w) \dots, \Psi_{m-1}(w)),\end{aligned} \tag{5.19}$$

where c is a constant determined by the initial conditions and $f, \Psi_1, \dots, \Psi_{m-1}$ are functions.

Let ψ denote the map which is the projection of z: $M_1 \to \mathbb{C}_1^{m+1}$ onto the complex Euclidean $(m-1)$-subspace $\mathbb{C}^{m-1}$ spanned by the last $m-1$ standard coordinate vectors $\epsilon_3, \dots, \epsilon_{m+1}$ of $\mathbb{C}_1^{m+1}$. Then we have

$$\psi_*(E_3 - E_4) = \text{proj}\,(z_*(E_3 - E_4)) = 0,$$

which follows from the fact that $Z = E_3 - E_4$ is constant along M_1 and

$$Z(s, 0, w) = Z_0 = (i, i, 0, \dots, 0).$$

Since $z_*(E_2) = iz_*(E_1)$, we have $i\psi_*(E_1) = \psi_*(E_2)$. Thus, the image $\psi(M_1)$ is a complex curve in $\mathbb{C}^{m-1}$.

Since $z_s = \tilde{\nabla}_{E_3-E_4} z = E_3 - E_4 = (i, i, 0, \dots, 0)$ on M_1 (with $t = 0$), (5.19) yields $\partial f/\partial s = 1$. Thus

$$f(s,w) = s + f_1(w) \tag{5.20}$$

for some complex-valued function $f_1 = f_1(w)$.

On the other hand, since $z(M_1)$ lies in the anti-de Sitter space time $H_1^{2m+1}(-1)$, (5.19) implies

$$2i(f\bar{c} - \bar{f}c) = 1 + \Psi\bar{\Psi},$$

where $\Psi\bar{\Psi} = \Psi_1\bar{\Psi}_1 + \cdots + \Psi_{m-1}\bar{\Psi}_{m-1}$. Therefore, (5.20) yields

$$2i\left(s(\bar{c} - c) + (f_1\bar{c} - \bar{f}_1 c)\right) = 1 + \Psi\bar{\Psi},$$

which implies that $\bar{c} = c$, i.e., c is a real number, and

$$2ic(f_1 - \bar{f}_1) = 1 + \Psi\bar{\Psi}. \tag{5.21}$$

Hence

$$f(s,w) = s + k(w) - \frac{i}{4c}(1 + \Psi\bar{\Psi}), \tag{5.22}$$

where $k = k(w)$ is a real-valued function. Consequently, we obtain

$$\begin{aligned} z(s,0,w) = \Big(&c + \frac{1}{4c}(1 + \Psi\bar{\Psi}) + i(s + k(w)), \\ &-c + \frac{1}{4c}(1 + \Psi\bar{\Psi}) + i(s + k(w)), \Psi(w)\Big). \end{aligned} \tag{5.23}$$

Since $z_t = z_*(E_3)$, (5.15-i) implies $z_{tt} = 2iz_t + z$. Solving this differential equation yields

$$z = (A_0 + tA_1)e^{it}, \tag{5.24}$$

where A_0, A_1 are constant vectors. From (5.23) and (5.24) we get

$$A_0 = z(s,0,w), \quad z_t(s,0,w) = iA_0 + A_1. \tag{5.25}$$

On the other hand, since

$$iA_0 + A_1 = z_t = E_3 = iz + (i, i, 0, \dots, 0) \tag{5.26}$$

at $t = 0$, (5.23), (5.25) and (5.26) yield $A_1 = (i, i, 0, \dots, 0)$. Therefore, (5.23), (5.24) and (5.25) imply

$$\begin{aligned} z(s,t,w) = \Big(&c + \frac{1}{4c}(1 + \Psi\bar{\Psi}) + i(s + t + k(w)) \\ &-c + \frac{1}{4c}(1 + \Psi\bar{\Psi}) + i(s + t + k(w)), \Psi(w)\Big)e^{it}. \end{aligned} \tag{5.27}$$

If we regard $s+t+k(w)$ as a new variable and denote it by u, then (5.27) yields

$$z(s,t,w) = \Big(c + \frac{1}{4c}(1+\Psi\bar{\Psi}) + ui, -c + \frac{1}{4c}(1+\Psi\bar{\Psi}) + ui, \Psi(w)\Big)e^{it}. \tag{5.28}$$

By choosing the initial conditions $z(0,0,0) = (-1,0,\ldots,0)$, we obtain from (5.28) that $c = -\frac{1}{2}$. Consequently, we obtain (3.5) from (5.28). Since z is an immersion, (5.28) implies that $\Psi'(w)$ is nowhere zero. This completes the proof of the theorem.

References

[1] A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publ., Dordrecht, 1986.

[2] B.-Y. Chen, *CR-submanifolds of a Kaehler manifold, I, II*, Journal of Differential Geometry **16** (1981), 305–322; **16** (1981), 493–509.

[3] B.-Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Archiv der Mathematik **60** (1993), 568–578.

[4] B.-Y. Chen, *A general inequality for submanifolds in complex-space-forms and its applications*, Archiv der Mathematik **67** (1996), 519–528.

[5] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Totally real submanifolds of $\mathbb{C}P^n$ satisfying a basic equality*, Archiv der Mathematik **63** (1994), 553–564.

[6] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *An exotic totally real minimal immersion of S^3 in $\mathbb{C}P^3$ and its characterization*, Proceedings of the Royal Society of Edinburgh, Section A **126** (1996), 153–165.

[7] B.-Y. Chen, G. D. Ludden and S. Montiel, *Real submanifolds of a Kaehler manifold*, Algebras, Groups and Geometries **1** (1984), 176–212.

[8] B.-Y. Chen and B.-Q. Wu, *Mixed foliate CR-submanifolds in a complex hyperbolic space are non-proper*, International Journal of Mathematics and Mathematical Sciences **11** (1988), 507–516.

[9] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geometriae Dedicata **20** (1986), 245–261.